The Recovery of Local Volatilities with Regularization in Chinese Markets

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1 Introduction

This document examines how to recover the local volatility surface of Shanghai Stock Exchange (SSE) 50 ETF options in Chinese markets.

The local volatility is a functional \( \sigma(\cdot, \cdot) \) such that under the physical measure \( \mathbb{P} \), the underlying asset price \( S_t \) is governed by the process:

\[
\frac{dS_t}{S_t} = \mu dt + \sigma(S_t, t)dW_t. \tag{1.1}
\]

A considerable amount of literature attempt to imply the local volatilities through Dupire’s equation. It states that the European vanilla call option price \( V(K, T) \), with strike \( K \) and maturity \( T \), satisfies the following equation:

\[
\begin{cases}
V_T - \frac{1}{2} \sigma^2(K, T) K^2 V_{KK} + (r - q) K V_K + q V = 0 \\
V\big|_{T=0} = (S_0 - K)^+
\end{cases} \tag{1.2}
\]

where \( S_0 \) is the current underlying price and \((K, T) \in (0, \infty) \times (0, T^*)\).

If the functional \( \sigma(\cdot, \cdot) \) is known, from (1.2) one can solve the option premium \( V(K, T) \) directly. This is called the direct problem. Conversely, if the continuous call option values \( V^*(K, T) \) are available, it is possible to imply the local volatilities through the observed data. Such issue is referred to as the inverse problem. A natural idea to solve it is to invert the PDE in (1.2) by substituting \( V^*(K, T) \) for \( V(K, T) \) as below:

\[
\sigma(K, T) = \sqrt{\frac{V^*_T + (r - q) K V^*_K + q V^*}{\frac{1}{2} K^2 V^*_{KK}}} . \tag{1.3}
\]

This inversion nevertheless has two drawbacks: data sparsity and ill-posedness.

First, the Dupire’s model assumes the data points \( V^*(K, T) \) are continuously available in \((K, T) \in (0, \infty) \times (0, T^*)\). This is, to some extent, sensible for S&P500 index options that typically have numerous market observations. The strike increments of these options are as small as 5 points. On the other hand, the tenors can extend from a week or less, to up to 12 months (and even to 60 months). For example, Cboe in [3] applied more than 250 data points. In China, however, data sparsity is a salient feature of SSE 50 options. There are only four maturities (current month, next month and the following two consecutive quarters) with up to sixteen strikes in each tenor (in 2018). The discrete observations raise difficulties in the application of Dupire’s model.

To address this issue, we assume SSE 50 ETF price follows the lognomal-mixture model. The parameters can be calibrated from the sparse market data. Then from the explicit formula we have the ability to obtain a dense option price surface with respect to different strikes \( K_i \) and maturities \( T_j \) (see [1] and [14]). The local volatilities in this documentation are based

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on this lognormal-mixture distributed price surface. Our following discussion presumes that a collection of continuous option prices are provided.

The second issue is more critical. The inverse problem is known to be ill-posed. The derivatives in (1.3), $V_K$, $V_{KK}$ and $V_T$ are subject to errors. When the numerator and denominator are small enough, especially for deep-in- or deep-out-of-money options with short expiry, the errors may dominate the true value. To illustrate this, one can investigate a hypothetical example:

**Example 1.** Suppose the dynamics of the underlier follows (1.1). The true local volatility is a constant $\sigma(K, T) = 0.2$ for all $(K, T) \in (0, \infty) \times (0, \frac{9}{365})$. Essentially, the call option price is congruous with the Black-Scholes model so that we can get the “exact” option prices and corresponding derivatives. When $K = 1.5$, $S_0 = 2.0$, $r = 0.03$ and $q = 0.03$, the recovered local volatilities in Python 3 are reported in the last column in Table 1:

<table>
<thead>
<tr>
<th>Tenor (days)</th>
<th>$V$</th>
<th>$V_K$</th>
<th>$V_{KK}$</th>
<th>$V_T$</th>
<th>$\sigma(K, T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>0.49938394</td>
<td>-0.99876788</td>
<td>8.8345 \times 10^{-11}</td>
<td>-0.01498152</td>
<td>20.00%</td>
</tr>
<tr>
<td>13</td>
<td>0.49946004</td>
<td>-0.99893208</td>
<td>1.9743 \times 10^{-12}</td>
<td>-0.01498398</td>
<td>20.00%</td>
</tr>
<tr>
<td>11</td>
<td>0.49954815</td>
<td>-0.99909630</td>
<td>1.0992 \times 10^{-14}</td>
<td>-0.01498644</td>
<td>19.95%</td>
</tr>
<tr>
<td>10</td>
<td>0.49958921</td>
<td>-0.99917842</td>
<td>3.6997 \times 10^{-16}</td>
<td>-0.01498768</td>
<td>18.26%</td>
</tr>
<tr>
<td>9</td>
<td>0.49963027</td>
<td>-0.99926055</td>
<td>5.8748 \times 10^{-18}</td>
<td>-0.01498891</td>
<td>nan</td>
</tr>
</tbody>
</table>

Table 1: The effect of varying maturities: the “exact” option values and derivatives generated from Black-Scholes formula, and the recovered local volatilities from (1.3).

As we can see in Table 1, even if the option values and its derivatives are generated from explicit formula rather than the numerical approximation (like the grid method), when the maturities approach zero, the errors tend to increase, and finally distort the results. In particular, with $T = \frac{9}{365}$, the denominator in (1.3) is indistinguishable from zero and the numerator is even negative, reporting erroneous volatility. From this example, we can conjecture that more severely deteriorating results can be acquired if we adopt finite difference method rather than explicit formula when $T \to 0$.

From the other perspective, the errors are also under the influence of the moneyness, as is shown in the following example. With extreme strike values, the recovered local volatilities lose their precision. Though in the last three rows (first two rows) of Table 2, the strikes seem to be excessively high (low) and unlikely to be observed in the market, it may still be possible to introduce large perturbations for lower (higher) strikes when the some numerical approach is employed. Finite difference method, for instance, may result in inaccurate values easily.

**Example 2.** Suppose the dynamics of the underlier follows (1.1). The true local volatility is a constant $\sigma(K, T) = 0.2$ for all $(K, T) \in (0, \infty) \times (0, \frac{9}{365})$. When $T = \frac{9}{365}$, $S_0 = 2.0$, $r = 0.03$ and $q = 0.03$, the recovered local volatilities in Python 3 are reported in the last column in Table 2:

<table>
<thead>
<tr>
<th>$K$</th>
<th>$V$</th>
<th>$V_K$</th>
<th>$V_{KK}$</th>
<th>$V_T$</th>
<th>$\sigma(K, T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.50</td>
<td>0.49963027</td>
<td>-0.99926055</td>
<td>5.8748 \times 10^{-18}</td>
<td>-0.01498890</td>
<td>nan</td>
</tr>
<tr>
<td>1.52</td>
<td>0.47964506</td>
<td>-0.99926055</td>
<td>2.5094 \times 10^{-16}</td>
<td>-0.01438935</td>
<td>26.80%</td>
</tr>
<tr>
<td>2.00</td>
<td>0.02503835</td>
<td>-0.49337069</td>
<td>6.346005584064</td>
<td>0.50692930</td>
<td>20.00%</td>
</tr>
<tr>
<td>3.00</td>
<td>1.2 \times 10^{-40}</td>
<td>-1.6 \times 10^{-38}</td>
<td>2.2035 \times 10^{-36}</td>
<td>4.0 \times 10^{-37}</td>
<td>20.00%</td>
</tr>
<tr>
<td>6.70</td>
<td>0.00000000</td>
<td>1.63 \times 10^{-322}</td>
<td>1.63 \times 10^{-322}</td>
<td>1.4 \times 10^{-322}</td>
<td>19.78%</td>
</tr>
<tr>
<td>6.71</td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>2.5 \times 10^{-323}</td>
<td>1.5 \times 10^{-323}</td>
<td>16.29%</td>
</tr>
<tr>
<td>6.72</td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>-0.00000000</td>
<td>16.29%</td>
</tr>
</tbody>
</table>

Table 2: The effect of varying moneyness: the “exact” option values and derivatives generated from Black-Scholes formula, and the recovered local volatilities from (1.3).

We resolve the ill-posedness of the inverse problem partially with regularization. Regularization introduces constraints or penalties to the original problem. The errors are suppressed by the introduced regularized information. Our application of regularization to local volatility surface is a hybrid of the PDE system framework of [9], [12] and [13], as well as [5] and [15], and the PDE linearization framework of [2], [6] and [11].
The rest of the document is organized as follows. Section 2 formulates the local volatility inverse problem with regularization from two perspectives. Section 3 clarifies the algorithm and performance in solving Problem 1 with regularized PDE system, and outlines the solution with the regularized linearization in reproducing kernel Hilbert space; the procedures of solving Problem 2 is also presented. The last section specifies the implementation on the recovery of Chinese SSE 50 ETF local volatility surface in RMI online dashboard.

2 The Inverse Problem

Suppose the continuous market observations are available at $T \in [T_1, T_N]$. Replacing $T^*$ with $T_N$, the inverse problem solving the local volatility is divided into Problem 1 and Problem 2. Problem 1 solves the time-independent local volatility $\sigma(K)$ at $(K, T) \in (0, \infty) \times (0, T_1)$. With this result, setting Problem 2 initial condition $\sigma(K, T_1) := \sigma(K)$, Problem 2 solves the time-dependent local volatility $\sigma(K, T)$ at $(K, T) \in (0, \infty) \times (T_1, T_N]$.

2.1 PDE system with regularization

Problem 1 in [12] was transformed as follows:

**Problem 1.** Let $y = \ln \frac{K}{S_0}$ and $U = \frac{V}{S_0} e^{qT}$. Find $a(y) := \frac{1}{2} \sigma^2(K)$, such that

$$J(a) = \inf_{\bar{a} \in A} J(\bar{a})$$

where for some $\delta > 0$,

$$J(\bar{a}) = \frac{1}{2} \|U(\cdot, T_1) - U^*(\cdot, T_1)\|^2_{L^2(R)} + \frac{\delta}{2} \|\bar{a}_y(\cdot)\|^2_{L^2(R)},$$ (2.1)

and $U^*(\cdot, T_1)$ is the transformed market observation.

The second term to the right of equal sign in (2.1) is the regularization term. It refrains the intensity of this regularization control. Extreme caution must be exercised in the selection of $\delta$. A excessively large value of $\delta$ will not only suppress the rapid variations of $a(y)$, but also flatten the local volatility curve. On the contrary, a small value that is close to zero may preserve the shape of $a(y)$ in some sub-interval within $y \in (-\infty, \infty)$, but fail to thwart the errors and lead to the radically varying $a(y)$ in other regions. In other words, an inappropriate $\delta$ will either distort the true recovered $a(y)$, or be dominated by the computation errors. The selection of $\delta$ is out of the scope in this paper.

Given proper $\delta$, by variational method and Dupire equation, the solution $a(y)$ in Problem 1 satisfies a nonlinear PDE system (2.2)-(2.4). The transformed call option value $U(y, T)$ satisfies

$$\begin{cases}
U_T - a(U_{yy} - U_y) + (r - q)U_y = 0 \\
U|_{T=0} = (1 - e^y)^+ \\
\Psi_T + (a\Psi)_y + (a\Psi)_y + (r - q)\Psi_y = 0 \\
\Psi|_{T=T_1} = U(y, T_1) - U^*(y, T_1)
\end{cases}$$ (2.2)

$$a_{yy} = \frac{1}{\delta} \int_0^{T_1} \Psi(\tau, y)(U_{yy} - U_y) d\tau$$ (2.3)

for $(y, T) \in (-\infty, \infty) \times (0, T_1]$.

$\sigma(K)$ results from the solution of (2.2)-(2.4). Once it is calibrated, the initial condition $a(y, T_1)$ for Problem 2 is set to be the value of $a(y)$, $y \in [0, T_1]$. As in [13], Problem 2 is expressed as:
**Problem 2.** Let \( y = \ln \frac{K}{S} \) and \( U = \frac{V}{S} e^{qT} \) as before, and define \( a(y, T)^n := \frac{1}{T^2} \sigma^2(K, T) \) for \( T_1 < \ldots < T_{n-1} \leq T \leq T_n < \ldots < T_N \). As of each interval \( (T_{n-1}, T_n) \), let \( h = T_n - T_{n-1} \) and suppose \( a^{n-1} \) is known, find the solution of \( a^n \), such that

\[
J_n(a^n) = \inf_{\overline{a}^n \in A_n} J_n(\overline{a}^n)
\]

where

\[
J_n(\overline{a}^n) = \frac{1}{2h} \left( \| U(\cdot, T_n) - U^*(\cdot, T_n) \|^2_{L^2(R)} + \frac{\sigma}{2} \left( \| \overline{a}^n_y \|^2_{L^2(R)} + \frac{1}{h} \| \overline{a}^n - \overline{a}^{n-1} \|^2_{L^2(R)} \right) \right).
\]

Assume \( a(y, T) = \frac{T-T_a}{h}a^n + \frac{T_a-T}{h}a^{n-1} \). When \( h \to 0 \), by variational method and Dupire equation, Problem 2 in the continuous case is reduced to the following PDE system:

\[
\begin{align*}
U_T - a(y, T)(U_{yy} - U_y) + (r - q)U_y &= 0 \\
U(y, T_1) &= U^*(y, T_1) \\
a_T - a_{yy} + \frac{1}{2\delta}(U - U^*)U_{yy} &= 0
\end{align*}
\]

for \((y, T) \in (-\infty, \infty) \times (T_1, T_N)\), and where \( U^*(y, T) \) is the transformed market observation.

Solving PDE system (2.6)–(2.7) iteratively produces the local volatility surface between \( T_1 \leq T \leq T_N \). Together with the result from Problem 1, the entire surface over \( T \in [0, T_N] \) is completed.

### 2.2 Linearization with regularization

[2], [6] and [11] solve the local volatilities from the other perspective. For reasons of space, linearization is introduced only for Problem 1 in this document. The linearization for Problem 2 can be found in [6].

The linearization technique decomposes the local volatilities such that \( a(y) = \frac{1}{2} \sigma_0^2 + f(y) \). \( \sigma_0 \) is a constant value close to the average magnitude of \( \sigma(K) \). \( f \) is \( C(\bar{\omega}) \)-small and \( f(y) = 0 \) outside \( \omega = (-b, b) \) for some positive number \( b \approx \frac{3}{2} T_1 \sigma_0^2 \).

In compliance with [11], we reformulate (2.2) as

\[
\begin{align*}
\ddot{U}_T - a(y)(\ddot{U}_{yy} - \ddot{U}_y) + (r - q)\ddot{U}_y + q\ddot{U} &= 0 \\
\ddot{U}|_{T=0} &= S_0(1 - e^y)^q
\end{align*}
\]

where \( \ddot{U}(y, T) = V(S_0 e^y, T) \). Along with the final condition \( \ddot{U}|_{T=T_1} = U^*(y, T_1) \), (2.8) may imply the functional \( a(\cdot) \).

With \( \sigma_0 \) as the initial constant guess, we have

\[
\begin{align*}
\ddot{U}_T - \frac{1}{2} \sigma_0^2 (\ddot{U}_{yy} - \ddot{U}_y) + (r - q)\ddot{U}_y + q\ddot{U} &= 0 \\
\ddot{U}|_{T=0} &= S_0(1 - e^y)^q
\end{align*}
\]

The solution of (2.9) is literally the Black-Scholes formula.

Let \( \ddot{U} = \dddot{U} + \xi + \nu \) and \( \nu \) is negligible small term. Impose the final condition at \( T = T_1 \). Then a intermediate variable \( \xi(y, T) \) satisfies

\[
\begin{align*}
\xi_T - \frac{1}{2} \sigma_0^2 (\xi_{yy} - \xi_y) + (r - q)\xi_y + q\xi &= a_0 f \\
\xi|_{T=0} &= 0 \\
\xi|_{T=T_1} &= U^*(y, T_1) - \dddot{U}(y, T_1)
\end{align*}
\] for

\[
\begin{align*}
a_0(y, T) &= \frac{S_0}{\sigma_0 \sqrt{2\pi t}} e^{-\frac{y^2}{2\sigma_0^2} + cy + dt} \text{ with } c = \frac{1}{2} + \frac{\mu}{\sigma_0^2} \text{ and } d = -\frac{\sigma_0^2}{2} \left( \frac{1}{2} + \frac{\mu}{\sigma_0^2} \right)^2 - q.
\end{align*}
\]

produces the Fredholm equation of the first kind:

\[
\xi(y, T_1) = \int_\omega K(x, y; T_1) f(y) dy, \quad x \in \omega
\]
where
\[
K(x, y; T_1) = \frac{S_0}{\sigma_0^2 \sqrt{\pi}} \int_{|x-y|+|y|}^{\infty} e^{-u^2} du = \frac{S_0}{2\sigma_0^2} \text{erfc} \left( \frac{|x-y|+|y|}{\sigma_0 \sqrt{2T_1}} \right).
\] (2.12)

Once \( f(y) \) is solved through (2.8) to (2.10), the local volatilities in \( T \in [0, T_1) \) is determined.

We apply trapezoidal rule to express the integral in order to obtain the approximation of \( f(y) \). In the discrete context, define the uniformly spaced mesh \( \vec{y} = (y_1, \ldots, y_I)^* \), and the corresponding vectors \( \vec{f} = (f(y_1), \ldots, f(y_I))^* \) as well as \( \vec{\xi} = (\xi(y_1, T_1), \ldots, \xi(y_I, T_1))^* \). The integral (2.11) can be expounded as
\[
\vec{\xi} = \textbf{K}\vec{f}
\] (2.13)

with the linear operator
\[
\textbf{K} = \begin{bmatrix}
K_{11} & \ldots & K_{1I} \\
\vdots & \ddots & \vdots \\
K_{I1} & \ldots & K_{II}
\end{bmatrix}.
\]

The solution of (2.13) is potentially of ill-posedness as the linear system may be subject to errors: \( \vec{\xi} = \textbf{K}\vec{f} + \epsilon \). To address this, We applied the modified Tikhonov regularization in reproducing kernel Hilbert space (RKHS). (see [16]). Express the bi-variate reproducing kernel function of a RKHS \( (\mathcal{H}) \) as \( R(\cdot, \cdot) \), and then (2.13) is regularized to solve:
\[
\tilde{J}(\vec{f}) = \inf_{\bar{f} \in \mathcal{H}} \tilde{J}(\bar{f})
\]
where for some \( \delta > 0, \)
\[
\tilde{J}(\bar{f}) = \|\textbf{K}\bar{f} - \vec{\xi}\|^2_{L^2(R)} + \delta \|\bar{f}\|_{\mathcal{H}}
\] (2.14)

The solution to (2.14) will be detailed in Section 3.4.

3 Solution to the Problems

The local volatility surface can be obtained by expressing Problem 1 and Problem 2 as the formulation in the previous section. Especially, Problem 2 can be well solved only if Problem 1 provides a decent initial value for \( T = T_1 \). However, solving Problem 1 is by no means an easy task. This section will focus on the attempted methods to resolve it. The procedures to solve Problem 2 will be outlined in the last part.

3.1 Two regularized PDE algorithms for Problem 1

The regularized PDE system can be established to solve Problem 1. The three test cases from [12] in Problem 1, based on the algorithm from [5] and [15], have been successfully recovered (see Figure 1). In contrast, the algorithm originally specified in [12] does not work well. Both algorithms are compared in Alg.3.

Solving PDE system (2.2) – (2.4) requires the discretization of (2.4) into:
\[
\frac{a_i^{(k)} - a_i^{(k-1)}}{\kappa} - \frac{a_{i-1}^{(k)} - 2a_i^{(k)} + a_{i+1}^{(k)}}{\Delta y^2} + g_i^{(k-1)} = 0
\]
or
\[
a_i^{(k)} = a_i^{(k-1)} + \kappa \left( \frac{a_{i-1}^{(k)} - 2a_i^{(k)} + a_{i+1}^{(k)}}{\Delta y^2} - g_i^{(k-1)} \right).
\] (3.1)

5
Figure 1: Recovery of three local volatilities in [12] with $q = 0, r = 0.12, S_0 = 10, T_1 = 1$.

where $g_i^{(k)}$ is the discretized expression of

$$g(y) = \frac{1}{\delta} \int_0^{T_1} \Psi(\tau, y)(U_{yy} - U_y)d\tau$$

in (2.4) for the k-th iteration. The additional $(a_i^{(k)} - a_i^{(k-1)})/\kappa$ serves as a "relaxed" term for the numerical scheme.

The successful algorithm Alg. 3a interprets (3.1) as the steepest descent search method for the entire PDE system, with the expression $a_i^{(k)} - 2a_i^{(k)} + a_{i+1}^{(k)} - \frac{g_i^{(k)}}{\Delta y}$ being the Fréchet derivative. Alg. 3b in [12], however, treat (3.1) as an ODE, because it seems to use a "parabolic equation" to approximate a "elliptic equation" if the nonlinearity in $g_i$ is overlooked. Arguably, this originates the significant differences between these two algorithms associated with their performances.

3.2 Limitations of the regularized PDE system

Though the test cases in [12] are replicated, one may find the drawbacks of Alg. 3a in solving Problem 1 are nontrivial from the numerical experiments.

a) Slow convergence near the boundaries

With exact Dirichlet boundary condition, we found that the recovered local volatilities $\sigma(T, K)$ converge fast near $K = S_0$, but the convergence to the "true values" is extremely slow at boundaries. We can take "smile" local volatility in Figure 1b as an example. As is shown in Figure 2, the local volatilities move from the initial guess (the red dotted line) to the values after $k$ iteration (blue line) and finally to the "true values" (orange dashed line) taking more
(a) Successful algorithm

1: set $k = 1$, guess $a^{(k-1)} = a_0$;
2: with $a^{(k-1)}$, solve (2.2) for $U$ and (2.3) for $\Psi$;
3: with $U$ and $\Psi$, solve (3.1) for $a^{(k)}$;
4: while $\|a^{(k)} - a^{(k-1)}\| < \epsilon$, do
5: $k = k + 1$
6: repeat Line 2 and 3
7: end while
8: $a = a^{(k)}$

(b) Unsuccessful algorithm

1: set $m = 1$, guess $a^{(m-1)} = a_0$
2: with $a^{(m-1)}$, solve (2.2) for $U$ and (2.3) for $\Psi$;
3: with $U$ and $\Psi$ in Line 2, solve (3.1) for $a^{(k)}$ with $k = 1 \ldots 1000$
4: $a^{(m)} = a^{(k)}$
5: while $\|a^{(m)} - a^{(m-1)}\| < \epsilon$, do
6: $m = m + 1$
7: repeat from Line 2 and 4
8: end while
9: $a = a^{(m)}$

Alg. 3: Two algorithms

than 155740 steps. A poor initial guess may make the computation time-consuming. This slow convergence also prevents us from setting large $\epsilon$ in Alg. 3a, because the early termination of the algorithm brings about inaccuracy when $K \ll S_0$ and $K \gg S_0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The convergence of the "smile" local volatilities}
\end{figure}

b) Sensitive regularization parameter $\delta$ and search step size $\kappa$

The $\delta$ in (2.4) stands for the "penalty" parameter in the regularization approach. The $\kappa$ in (3.1) represents the line search step size. The former is hard to be determined as an excessively large one may over-regularize the objective function (2.1) and an inappropriate small one may under-
regularize it. Even a "proper" $\delta$ is given, an excessive large $\kappa$ may amplify the errors, making the algorithm divergent; a smaller but still improperly large one may lead to a convergent result. However, it will be inaccurate at the boundaries. Figure 3 is an example of this, with $\kappa$ being 10 times as large as the that in Figure 1b. It seems that $\kappa$ is too large to fit the "true value". In contrast, a drastically small one makes the convergence extremely slow. One tricky technique is to set $\kappa$ proportional to the regularization parameter $\delta$, but the ratio cannot be easily determined.

![Figure 3: Search step size $\kappa$ 10 times as large as that in Figure 1b](image)

c) **Unknown boundaries**

The examples above presume known boundary conditions for both $K \to 0$ and $K \to \infty$. Now we change this assumption, making boundaries different from the exact ones, in order to investigate to what extent the local volatilities can be recovered.

We scale the left and right boundaries in Figure 1 with factors 0.9 and 1.1, respectively. As we can see from the Figure 4 and 5, perturbed boundary conditions may not affect the shape of local volatilities near $K = S$, but the recovered local volatilities near boundaries are distorted.

d) **Short expiry and small local volatility level**

V. Isakov [11] proposes a linearization model whose effective recovered "range" is approximately:

$$y \in (-b, b), \text{ with } b \approx \frac{3}{2} \sigma_0^2 T_1.$$  \hspace{1cm} (3.2)

To put it another way, the uniqueness of the solution in V. Isakov [11] can only be proved within $y \in (-b, b)$. The local volatilities outside this region are rarely recovered. As $T_1 \to 0$, $b$ will shrink to zero linearly; as the level of $\sigma(y)$, or $\sigma_0$, tends to zero, $b$ will diminish in a quadratic order, and the "range" of $y$ will degenerate to $y = 0$ quickly.

The regularized PDE system cannot make a breakthrough in this limited recovery. In Figure 1, 3, 4 and Figure 5, only the "true" local volatilities within the green-dashed vertical lines ($y = \pm b$) can be mostly recovered. This implies the recovered regions of Alg. 3a are also restricted to (3.2) in some sense.

To show this further, we have the following three sets of examples, each of whom consists of two "sine" local volatility functions. We set these drastic varying "sine" shapes to check the robustness of Alg. 3a and view the inherent length of the recovered "range" of each case. The first set is with parameter $\sigma_0 = 1, T_1 = 0.7$ (Figure 6), second set with $\sigma_0 = 1, T_1 = 0.1$ (Figure 7), and the third set with $\sigma_0 = 0.1, T_1 = 0.1$ (Figure 8). The exact boundary conditions are given.
(a) "Flat" local volatilities

(b) "Smile" local volatilities

(c) "Skew" local volatilities

Figure 4: The local volatilities with 90\% scaled boundaries

(a) "Flat" local volatilities

(b) "Smile" local volatilities

(c) "Skew" local volatilities

Figure 5: The local volatilities with 110\% scaled boundaries
Figure 6: “Sine” local volatilities: \(\sigma_0 = 1, T = 0.1\)
\(S_0 = 20, r = 0.05, q = 0.05\)

Figure 7: “Sine” local volatilities: \(\sigma_0 = 1, T = 0.7\)
\(S_0 = 20, r = 0.05, q = 0.05\)

Figure 8: “Sine” local volatilities: \(\sigma_0 = 0.1, T = 0.1\)
\(S_0 = 20, r = 0.05, q = 0.05\)

Figure 6 to 8 illustrate the “range” Alg. 3a can recover under three sets of parameters. The cases from top to bottom tend to be “narrower” despite exact boundaries are given. In the real world, \(\sigma_0\) and \(T_1\) can be as small as 0.1 and 0.02, respectively. For example, the market strikes \(K \in (2.2, 2.55)\) of Chinese 50 ETF options on March 13, 2017 with short expiry \(T_1 = 0.025\) cannot be covered by 3.2 with \(b \approx 0.031\), or accordingly \(K \in (2.283, 2.430)\). That means only a diminutive portion of local volatilities can be effectively implied.

Similar results on this limited “range” can be found in [8]. A heuristic interpretation may be helpful to understand this. If \(T_1 = 0\), the local volatilities play no roles in option pricing, as the option prices are strictly \((S_0 - K)^+\). When \(T_1 \in (0, \epsilon)\), the prices are still approximately piecewise linear except for the at-the-money point, if \(\sigma_0\) is not sufficiently large. The local volatility \(a(y)\) is relevant to the convexity of the option prices with respect to \(y\). Hence the explanatory power of \(a(y)\) in the non-at-the-money region is so slim that the numerical scheme becomes invalid. In other words, given a short expiry, the fast decaying \(U_{yy} - U_y\) term in (2.2)
is approximately the product of $e^y$ and the Dirac delta function at $y = 0$. The importance of $a(y)$ thus decreases if the option is not at-the-money. Portentially, $a(y)$ is numerically not recoverable with small $T_1$ and $\sigma_0$. The region that cannot be recovered near boundaries is called the “insignificant region” in [4]. (See Figure 9.)

![Figure 9: The insignificant region in terms of pricing from [4.](#) e) Poor performance in market data

Due to the problems listed from 3.2.a to 3.2.d, the performance of regularized PDE system on the real market data is not satisfactory. Determining the shape of non-at-the-money local volatilities is of considerable difficulty. Also, the put-call parity is frequently violated. The percentage errors $\frac{|V - V^*|}{V^*}$ for some observation points are remarkable, especially for the out-of-money options.

We have implemented a test on the SSE 50 ETF option market data on March 13, 2017. The percentage errors when $K < S_0$ for put options and $K > S_0$ for call options are remarkably big in Table 4.

<table>
<thead>
<tr>
<th></th>
<th>2.2</th>
<th>2.25</th>
<th>2.3</th>
<th>2.35</th>
<th>2.4</th>
<th>2.45</th>
<th>2.5</th>
<th>2.55</th>
</tr>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Call</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#1</td>
<td>0.1535</td>
<td>0.10385</td>
<td>0.05585</td>
<td>0.01655</td>
<td>0.00365</td>
<td>0.00105</td>
<td>0.00055</td>
<td>0.00035</td>
</tr>
<tr>
<td>#2</td>
<td>0.00107</td>
<td>0.00109</td>
<td>0.00173</td>
<td>0.00473</td>
<td>0.00152</td>
<td>-0.00002</td>
<td>-0.00040</td>
<td>-0.00033</td>
</tr>
<tr>
<td>#3</td>
<td>1%</td>
<td>1%</td>
<td>3%</td>
<td>29%</td>
<td>42%</td>
<td>-2%</td>
<td>-72%</td>
<td>-95%</td>
</tr>
<tr>
<td>Put</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#1</td>
<td>0.0004</td>
<td>0.00045</td>
<td>0.00195</td>
<td>0.01195</td>
<td>0.04835</td>
<td>0.0978</td>
<td>0.1467</td>
<td>0.19665</td>
</tr>
<tr>
<td>#2</td>
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<td>-0.00004</td>
<td>0.00106</td>
<td>0.00473</td>
<td>0.00219</td>
<td>-0.00144</td>
<td>-0.00125</td>
<td>-0.00137</td>
</tr>
<tr>
<td>#3</td>
<td>-81%</td>
<td>-9%</td>
<td>55%</td>
<td>40%</td>
<td>5%</td>
<td>-1%</td>
<td>-1%</td>
<td>-1%</td>
</tr>
</tbody>
</table>

Table 4: Market data with $S_0 = 2.355$, $T_1 = 0.0247$, $r = 0.0418$, $q = 0.0452$. #1, #2 and #3 representing $V^*$, $(V - V^*)$ and $\frac{|V - V^*|}{V^*}$, respectively.

3.3 Failed attempts

We have attempted to reformulate (2.1) in Problem 1 to overcome above limitations. One candidate is the following problem to optimize the implied volatilities:

**Problem 3.** Let $y = \ln \frac{K}{S_0}$ and $U = \frac{V}{S_0} e^{rT}$. Find $a := \frac{1}{2} \sigma^2(y)$, such that

$$J(a) = \inf_{\tilde{a} \in A} J(\tilde{a})$$

where

$$J(\tilde{a}) = \frac{1}{2} \left\| \sigma_{\text{imp}}(\cdot, T_1; \tilde{a}) - \sigma^*(\cdot, T_1) \right\|_{L^2(R)}^2 + \frac{\delta}{2} \left\| \tilde{a}_y(\cdot) \right\|_{L^2(R)}^2$$

(3.3)
By variational method and Dupire’s equation, the solution in Problem 3 satisfies a nonlinear PDE system (3.4)–(3.6). The transformed call option value \( U(y,T) \) satisfies

\[
\begin{align*}
\left\{ \begin{array}{l}
U_T - a(U_{yy} - U_y) + (r - q)U_y = 0 \\
U |_{T=0} = (1 - e^y)^+
\end{array} \right. \\
\left\{ \begin{array}{l}
\Psi_T + (a \Psi)_{yy} + (a \Psi)_y + (r - q) \Psi_y = 0 \\
\Psi |_{T=T_1} = (\sigma_{imp}(y,T_1) - \sigma^*_{imp}(y,T_1)) / \frac{\partial V}{\partial \sigma_{imp}}(y,T_1)
\end{array} \right.
\end{align*}
\]

(3.4) 

(3.5)

\[ a_{yy} = \frac{1}{\delta} \int_0^{T_1} \Psi(\tau,y)(U_{yy} - U_y)d\tau \]

(3.6)

for \((T, K) \in (0, T_1] \times (0, \infty)\). \( \sigma_{imp}(y,T_1) \) and \( \sigma^*_{imp}(y,T_1) \) represent the model and the market observed implied volatilities, respectively.

This reformulation also features significant ill-posedness. Note the denominator in (3.5) is the vega of the European vanilla option. In terms of the short maturity or low \( \sigma_0 \) as in Section 3.2.d, this vega in the denominator tends to be zero, making the numerical method for PDE system (3.4)–(3.6) ill-posed.

The other way to modify (2.1) is to replace constant \( \delta \) with \( \delta(y) \) such that it is spatially varying. This idea is brought from [10]. Different sections among \((-\infty, \infty)\) are regularized differently. However, the result improvement requires further investigation.

### 3.4 Solve Problem 1 with linearization in RKHS

Though the regularized linearization in Section 2.2 may not produce a superior result over the PDE system approach, it is less time-consuming. On the contrary, Alg. 3a contains iterations and is more computationally intensive. The effectiveness of linearization approach makes it more applicable.

The first step is to have an initial guess of the constant volatility \( \sigma_0 \) in (2.9). It can be achieved by assigning the value minimizing the \( \| U^*(\cdot, T_1) - \hat{U}(\cdot, T_1) \| \), or some other values nearby. \( \hat{U}(y_i, T_1) \) can be calculated fast by either solving (2.9) via finite difference method or applying Black-Scholes formula. Then to solve (2.14), we express the reproducing kernel of Hilbert space \( \mathcal{H} \) (see also [7]) as:

\[
R(x, y) = \frac{1}{2 \sinh 2b} ( \cosh(x + y) + \cosh(|x - y| - 2b) )
\]

The solution of \( \tilde{f} \) associated with \( \mathcal{H} \) is accordingly denoted in terms of \( \lambda_i \) such that

\[
\tilde{f}(y) = \sum_{i=1}^{I} \lambda_i R(y, y_i)
\]

(3.7)

Following [16], the regularized objective function in (2.14) can be expressed associated with \( \lambda_i \):

\[
\tilde{J}(\tilde{f}) = \left\| \sum_{i=1}^{I} \lambda_i K(R(\cdot, y_i)) - \xi \right\|_{L^2(R)}^2 + \delta \left\| \sum_{i=1}^{I} \lambda_i R(\cdot, y_i) \right\|_{\mathcal{H}}
\]

\[
= \sum_{i=1}^{I} \lambda_i K(R(\cdot, y_i)) - \xi \right\|_{L^2(R)}^2 + \delta \sum_{i=1}^{I} \sum_{j=1}^{I} \lambda_i \lambda_j R(y_i, y_j).
\]

(3.8)

The problem to imply \( \tilde{f} \) in (2.14) is thus equivalent to that to solve the optimal \( \tilde{\lambda} = \{\lambda_1, \ldots, \lambda_I\}^* \) from (3.8). It is literally the solution to the linear system:

\[
(A + \delta R)\tilde{\lambda} = \tilde{\eta}
\]
where
\[
[R]_{ij} = R(y_i, y_j), \quad [A]_{ij} = \int_{-b}^{b} \phi_i(x)\phi_j(x)dx, \quad \eta_i = \int_{-b}^{b} \phi_i(x)\xi(x)dx
\]
and
\[
\phi_i(y) = K(R(\cdot, y_i))(y) = \int_{-b}^{b} K(y, s)R(s, y_i)ds.
\]

The above integrals can be numerically evaluated with trapezoidal rule.

### 3.5 Solve Problem 2 with regularized PDE system

Once \(\sigma(K)\) over \([0, T]\) is recovered in Problem 1, it serves as the initial value in solving Problem 2, i.e., \(\sigma(K, T_1) = \sigma(K)\). Fully implicit scheme is employed in finite difference method. The maturity domain is partitioned into \(T_1 < \ldots < T_{n-1} \leq T \leq T_n < \ldots < T_N\). Each time step contains integrated iterations to solve (2.6) and (2.7). More specifically, for \(T = T_n\), denote \(a(y, T_n)\) and \(U(y, T_n)\) in \(j\)-th iteration as \(a_{i,n}^j\) and \(U_{i,n}^j\) at \((y, T_n)\), given solved \(a(y_i, T_{n-1})\) and \(U(y_i, T_{n-1})\). The initial volatility value of the first iteration \(j = 0\) is set as \(a_{i,n}^0 = a(y_i, T_{n-1})\). For \(j = 1, 2, \ldots\), we solve one step fully implicit scheme, first with known \(a_{i,n}^{j-1}\) for \(U_{i,n}^j\) in (3.9), and second with newly solved \(U_{i,n}^j\) for updated \(a_{i,n}^j\) in (3.10):

\[
\begin{align*}
\frac{U_{i,n}^j - U_{i,n}^{j-1}}{\Delta T} - a_{i,n}^{j-1} \Delta_{yy}U_{i,n}^j + (r - q + a_{i,n}^{j-1}) \Delta_y U_{i,n}^j &= 0 \\
a_{i,n}^j - a_{i,n}^{j-1} \Delta T &= \Delta_{yy}a_{i,n}^j + \frac{1}{2\delta}(U_{i,n}^j - U_i^*)\Delta_{yy}U_{i,n}^j = 0
\end{align*}
\]

with

\[
\Delta_y U_{i,n}^j = \begin{cases} 
\frac{U_{i,n+1} - U_{i,n}}{\Delta y}, & \text{if } r - q + a_{i,n}^j < 0 \\
\frac{U_{i,n}^j - U_{i,n-1}^j}{\Delta y}, & \text{otherwise},
\end{cases}
\]

\[
\Delta_{yy}U_{i,n}^j = \frac{U_{i,n+1}^j - 2U_{i,n}^j + U_{i,n-1}^j}{(\Delta y)^2},
\]

and

\[
\Delta_{yy}a_{i,n}^j = \frac{a_{i,n+1}^j - 2a_{i,n}^j + a_{i,n-1}^j}{(\Delta y)^2}.
\]

These process will continue until \(\|a_{i,n}^j - a_{i,n}^{j-1}\|\) approaches zero. After that, the forward recursion moves to solve \(a(y, T_{n+1})\). [13] also provides the finite element algorithm for Problem 2.

### 4 Implementation

In RMI application, Problem 1 is solved with regularized linearization in RKHS, since this method is of greater computational efficiency. Problem 2 is addressed with finite difference method for simplicity.

SSE 50 ETF local volatility surface is established in RMI online dashboard. The risk-free interest rate is set to be SHIBOR rate associated with the longest option maturity. The dividend yield is the averaged value across all the strikes with the longest maturity implied from the put-call parity. The continuous call option values in RMI model are interpolated mid-prices from the out-of-the-money options (see also [14]). Both the graphic and tabular views of the local volatilities on end-of-day SSE 50 ETF options are available from January 3, 2017 onward.
Figure 10: RMI local volatility surface on SSE 50 ETF options

References


