Documentation of Callable and Putable bonds

RMI Bond Team

1 Market Situation

1.1 Chinese Market

In Chinese market, there are totally 651 callable/putable bonds.

1.2 General Contractual Rules

Following work of [Dai and Xu (2014)], below statements describe general payoff rules of the bonds, according to their official issuance documents. Note that the initial coupon rate $C$, coupon payment dates $T_i$, maturity $T_n$ and redemption time $T_m$ may vary from sample to sample.

1. The bond has a face value $F$ and is matured at $T_n$;

2. Denote $t$ to be current time. A series of coupon dates $t < T_1 < T_2 < \ldots < T_m$ is prescribed. Coupon $C$ is paid on date $T_j$ for $1 \leq j \leq m$;

3. Coupon rate on each payment date may not always be $C$.

- For putable bonds, there are two cases: I. at issuance date, the issuer may set a fixed increasing amount $\delta$ of coupon rate after option time, if the bond is not redeemed. That is the coupon rate between issuance date and option date $T_m$ is $C$, and is $C + \delta$ from option date $T_m$ to maturity $T_n$. II. Right before the date $T_m$, the issuer could decide whether to increase the coupon in the future coupon dates $T_{m+1} < T_{m+2} < \ldots < T_n$ by certain amount $\delta \in [\delta_{min}, \delta_{max}]$ or not, and announce the decision to the holder.

- For callable bonds, there is just one case: on issuance date, the issuer may set a fixed amount $\delta$ of coupon rate after option time $T_m$, if the bond is not redeemed by the issuer.

Normally, we call $\delta$ coupon step-up. For convenience in our model, if issuer decide not to increase coupon, we simply set $\delta$ as 0;
4. Based on the coupon announcement,

- For putable bonds, the holder could choose to either redeem the bond on $T_m$ with price $K$ or hold it until maturity $T_n$. If the bond is not redeemed at $T_m$, the announced new coupon $C + \delta$ will be paid at $T_j$, $m + 1 \leq j \leq n$. At $T_n$ the face value will be paid and the bond expires.

- For callable bonds, the issuer could choose to either call the bond back on $T_m$ with price $K$ or keep it to maturity $T_n$. If the bond is not called back at $T_m$, the announced new coupon $C + \delta$ will be paid at $T_j$, $m + 1 \leq j \leq n$. At $T_n$ the face value will be paid and the bond expires.

Below is an timeline example of one callable/putable bond.

![Timeline Example]

2 Pricing Method

Following work of Dai and Xu (2014), we use Hull-White model to price callable and putable bonds.

2.1 Payoffs

To formulate the model, we first analyze cash flows generated by these bonds.

2.1.1 Putable Bond

For putable bond, since the coupons between $T_1$ and $T_m$ are pre-specified and fixed, they could be discounted according to the issuer’s credit rating. However, in order to determine the bond’s value at $T_m$, some analyses are necessary. In the following we assume the rationality of both issuer and holder and each scenario is at $T_m$.

1. If the interest rate was low, the bond’s value will exceed $K$ due to the high coupon rate. In this case, if the step-up coupon $\delta$ is not pre-specified by issuer on the issuance date, the issuer should remain the coupon at old level $C$ and the bond will not be redeemed.\footnote{Here we only consider the change in risk free bond market and do not consider the competitors since that would introduce intractable complications to the model. However, in reality, other companies may issue new bonds and these new issuance would certainly influence the old issuer’s decision.}
2. On the contrary, if the interest rate was high, this bond’s value will drop below $K$ hence the holder has the incentive to redeem. In this case, the issuer may decide to increase the future coupon to avoid redemption since redemption could potentially cause financing distress. This situation could be further divided into two cases:

(a) If the interest rate was so high that the issuer could not keep the value above $K$ by raising coupon (Recall that the coupon is bounded above by the terms), the holder would certainly redeem the bond at $K$.

(b) If the issuer could avoid the redemption by raising coupon, it is optimal for the issuer to make the bond value just above $K$, say, $K + \epsilon$, where $\epsilon$ is an arbitrarily small positive number. In this case, we could approximate the value by $K$.

Due to the above analysis, the value of the putable bond at $T_m$ is

$$V(T_m) = \max\{V^*(T_{m+1},...,n; C + \delta), K\}$$

$$= K + [V^*(T_{m+1},...,n; C + \delta) - K]^+, \quad (2.2)$$

where $V^*(T_{m+1},...,n; C + \delta)$ is the time $T_m$ value of a bond with coupon $C + \delta$ paid at $T_i, i = m + 1, \ldots, n$ and face value $F$ paid at $T_n$.

In light of (2.2), the bond value at $T_m$ is decomposed into two components. The first component is a zero coupon bond with payment $K$ at $T_m$ and it could be combined with the coupons at $T_i, i \leq m$ to form a coupon bearing straight bond. The second component is a call option on a coupon bearing bond, struck at $K$ and matured at $T_m$. We could separately (but consistently) price them and add their value together to obtain the value of the original putable bond.

### 2.1.2 Callable Bond

Similarly, for callable bond the coupons between $T_1$ and $T_m$ are pre-specified and fixed. Following are some analysis to determine the bond’s value at $T_m$. We assume the rationality of both issuer and holder.

1. If the interest rate was low, the bond’s value will exceed $K$ due to the high coupon rate. In this case the issuer has the incentive to redeem bonds and re-issue bonds with a low coupon rate.

---

2Since both the issuer and holder could take actions, there is essentially a game between them. However, the game here is simple if we note that the time is pre-specified, the sequence of action is predetermined and assume the rationality of both issuer and holder. In the case of callable bonds, in which the game could be played at any time and by any party, one could refer to Chen et.al. (2013).
2. On the contrary, if the interest rate was high, this bond’s value will drop below $K$. In this case, the issuer may not exercise the right of redemption.

Thus, the value of the callable bond at $T_m$ is

$$V(T_m) = \min\{V^*(T_{m+1},\ldots,n; C + \delta), K\}$$

(2.3)

$$= K + [K - V^*(T_{m+1},\ldots,n; C + \delta)]^+,$$

(2.4)

where $V^*(T_{m+1},\ldots,n; C)$ is the time $T_m$ value of a bond with coupon $C$ paid at $T_i, i = m + 1, \ldots, n$ and face value $F$ paid at $T_n$.

In light of (2.4), similar to analysis of putable bond, the callable bond value at $T_m$ is decomposed into two components. The first component is a zero coupon bond with payment $K$ at $T_m$ and it could be combined with the coupons at $T_i, i \leq m$ to form a coupon bearing straight bond. The second component is a put option on a coupon bearing bond, struck at $K$ and matured at $T_m$. We could separately (but consistently) price them and add their value together to obtain the value of the original callable bond.

### 2.2 Pricing the Straight Bond

In order to price the straight bond consistent with market, we discount the cash flows generated by the bonds according to suitable yield curve. Since cash $C + K I_{i=m}$ is paid at time $T_i, 1 \leq i \leq m$ the present value of the straight bond is

$$V_B(t) = \sum_{i=1}^{m} (C + K I_{i=m}) e^{-Y^d(t,T_i)(T_i-t)},$$

(2.5)

where $Y^d(t,T_i)$ is the continuous risky yield prevailing from $t$ to $T_i$, which could be extracted (or interpolated) from the reference yield curve.

### 2.3 Pricing the Embedded Option

In order to price the option with interest rate and credit risk, we choose the celebrated Hull-White short rate model and augment it by deterministic intensity model. These two model components are all easily calibrated to the market curves as discussed shortly below.

#### 2.3.1 Intensity model

We first introduce the models and we begin with the intensity model. Specifically, we use the deterministic version by assuming that the intensity of default $\lambda(t)$ is a deterministic

---

$I$ is the indicator function.
function of $t$ and calibrate it as follows. Note that the time $t$ price of a default free bond matured at $T$ is

$$P^\text{df}(t, T) = \mathbb{E}_t^Q \left\{ \exp \left\{ - \int_t^T r(u) du \right\} \right\}, \quad \text{(2.6)}$$

and that assuming zero recovery rate upon default\footnote{This assumption may not be very realistic. However, generally recovery rate is not observable in the market and this assumption makes the calculation much easier.} the time $t$ price of a defaultable bond matured at $T$ is

$$P^d(t, T) = \mathbb{E}_t^Q \left\{ \exp \left\{ - \int_t^T (r(u) + \lambda(u)) du \right\} \right\}, \quad \text{(2.7)}$$

where $Q$ is the risk neutral measure.\footnote{These results could be found in many standard textbooks on quantitative finance, e.g., Brigo and Mercurio (2006).}

Since $\lambda(t)$ is assumed to be non-random, we have

$$\exp \left\{ \int_t^T \lambda(u) du \right\} = \frac{P^\text{df}(t, T)}{P^d(t, T)} = \exp \left\{ (Y^d(t, T) - Y^\text{df}(t, T))(T - t) \right\}, \quad \text{(2.8)}$$

where $Y^d(t, T)$ and $Y^\text{df}(t, T)$ are the continuous yield of defaultable and default free bond between $t$ and $T$, respectively. Therefore,

$$\int_t^T \lambda(u) du = (Y^d(t, T) - Y^\text{df}(t, T))(T - t). \quad \text{(2.9)}$$

We also assume that $\lambda(t)$ is continuous and by differentiating with respect to $T$ in (2.9), we obtain the following formula for $\lambda(t)$ in terms of the credit spread:

$$\lambda(T) = \frac{\partial}{\partial T} \left[ (Y^d(t, T) - Y^\text{df}(t, T))(T - t) \right], \quad \text{(2.10)}$$

where we have assumed the smoothness of the yield curve.

### 2.3.2 Hull-White model

We use Hull-White model to price bond embedded option. The risk neutral dynamics of default free short rate $r(t)$ is assumed to be

$$dr(t) = [\vartheta(t) - ar(t)] dt + \sigma dW(t), \quad r(0) = r_0,$$

where $W(t)$ is a standard one dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}_t, Q)$, $\sigma$ is the instantaneous volatility of $r(t)$, and $\vartheta(t)$ is a deterministic function.

There are totally three parameters in Hull-White model: $\vartheta(t), a, \sigma$. The parameter’s calibration methods are as follows.
• Instantaneous forward rate curve \( f(t, T) \), \( T \geq t \) could be obtained by

\[
f(t, T) = -\frac{\partial \ln P^d(t, T)}{\partial T} = Y^d(t, T) + (T - t) \frac{\partial Y^d}{\partial T}(t, T). \tag{2.11}
\]

Then \( \vartheta(t) \) is determined by

\[
\vartheta(s) = \partial_T f(t, s) + a f(t, s) + \frac{\sigma^2}{2a} (1 - e^{-2a(s-t)}). \tag{2.12}
\]

• We note that the determination of \( \sigma \) and \( a \) in the Hull-White model requires extra products. Unfortunately, extra calibration products are generally not available in current Chinese market. Due to this restriction, we resort to econometric approach proposed in [Meng et al., 2013]. In their method, \((\sigma, a)\) is estimated through minimizing the following objective function:

\[
g(a, \sigma) = \sum_j \frac{1}{M_j^2} \left( \text{std} \left( \log(P_{t_i, t_i+M_j}) - \log \left( \frac{P_{t_i-1, t_i+M_j}}{P_{t_i-1, t_i}} \right) \right) - \sqrt{\text{Var}M_j} \right)^2 \tag{2.13}
\]

subject to \( a > 0, \sigma > 0 \), where

- \( M_j \): the spectrum of maturities of the yields;
- \( P_{t_i,s} \): the time \( t \) price of default free zero coupon bond matured at \( s \);
- \( \text{std} \): the standard deviation operator with \( i \) runs over the sampling period;

and

\[
\text{Var}M_j = \frac{1}{a^2} \left( 1 - e^{-aM_j} \right)^2 \cdot \frac{\sigma^2}{2a} \left( 1 - e^{-2a\Delta t} \right),
\]

where \( \Delta t \) is the time step between observations, in our case, \( \Delta t = \frac{1}{250} \).

### 2.3.3 Derivation and Solution of PDEs

Given the intensity function \( \lambda(t) \) and the dynamics of \( r(t) \), the embedded option could be priced via partial differential equations (PDEs). We first take note of the following Lemma:

**Lemma 2.1** Suppose that for \( s_1 \leq t \leq s_2 \),

\[
V(r_t, t) = \mathbb{E}^Q \left[ \exp \left( - \int_{t}^{s_2} (r_u + d(u)) \, du \right) h(r_t) \bigg| F_t \right], \tag{2.14}
\]

where \( Q \) is risk-neutral measure, \( r_t \) is the only random variables and follows Hull-White model \((dr_t = (\vartheta(t) - ar_t)dt + \sigma dE_t), \) \( d \) is deterministic time-dependent, and \( h(r) \) is given. Then

\[
\frac{\partial V}{\partial t} + (\vartheta(t) - ar) \frac{\partial V}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial r^2} - (r + d(t)) V = 0 \tag{2.15}
\]

for \( s_1 \leq t \leq s_2 \) with boundary conditions \( V(r_{s_2}, T) = h(r_{s_2}) \).
In our case, the underlying bond pays coupon \( C + \delta + FI_{j=n} \) at \( T_j \), \( m + 1 \leq j \leq n \). Lemma 2 shows that the value \( P \) of the underlying at time \( T_m \) can be found by solving the following PDE

\[
\begin{aligned}
\frac{\partial P}{\partial t} + (\vartheta(t) - ar)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} - (r + \lambda(t))P &= 0, \\
P(T_j-, r) &= P(T_j, r) + C + \delta, \\
P(T_n, r) &= C + \delta + F.
\end{aligned}
\]  

(2.16)

With the underlying price \( P(T_m, r) \), the price of the call option, as a function of \((t, r)\), would satisfy the PDE

\[
\begin{aligned}
\frac{\partial O}{\partial t} + (\vartheta(t) - ar)\frac{\partial O}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 O}{\partial r^2} - (r + \lambda(t))O &= 0, t \in [0, T_m), \\
O(T_m, r) &= (P(T_m, r) - K)^+. 
\end{aligned}
\]  

(2.17)

Since the \( r(t) \) in Hull-White model is normally distributed, the spacial solution domain for the above PDEs is \( r \in (-\infty, \infty) \).

We use Euler explicit numerical method to solve this PDE. First we discretize the space and time dimension: \( \Delta r = (r_{max} - r_{min})/N \) and \( \Delta t = (T_{max} - T_{min})/M \). In our implementation, we set \( r_{min} = -0.15 \), \( r_{max} = 0.15 \), \( N = 1600 \), \( \Delta t = 0.01 \). Then \( r_i = r_{min} + i\Delta r, \ i = 0 \cdots N \) and \( t_j = T_{max} - j\Delta t, \ j = 0 \cdots M \).

After discretizing the space dimension and time dimension we can get following equation:

\[
\frac{P_{i+1}^j - P_i^j}{\Delta t} + A_1\left(\frac{P_{i+1}^j - P_i^j}{\Delta r}I_{\{A_1<0\}} + \frac{P_i^j - P_{i-1}^j}{\Delta r}I_{\{A_1>0\}}\right) + A_2\frac{P_{i+1}^j - 2P_i^j + P_{i-1}^j}{\Delta r^2} + A_3P_i^j = 0,
\]

where

\[
\begin{aligned}
A_1 &= \vartheta^j - ar_i, \\
A_2 &= \frac{1}{2}\sigma^2, \\
A_3 &= r_i + \lambda^j.
\end{aligned}
\]

Arrange the above equation we can get

\[
\frac{P_{i+1}^j}{\Delta t} = B_{i-1}^jP_{i-1}^j + C_i^jP_i^j + D_{i+1}^jP_{i+1}^j,
\]

where

\[
\begin{aligned}
B_{i-1}^j &= \left(\frac{A_1I_{\{A_1>0\}}}{\Delta r} - \frac{A_2}{\Delta r^2}\right), \\
C_i^j &= \left(\frac{1}{\Delta t} - \frac{A_1}{\Delta r}\left(I_{\{A_1<0\}} - I_{\{A_1>0\}}\right) + \frac{2A_2}{\Delta r^2} - A_3\right), \\
D_{i+1}^j &= \left(-\frac{A_1I_{\{A_1>0\}}}{\Delta r} + \frac{A_2}{\Delta r^2}\right).
\end{aligned}
\]
Thus we have a linear system at every time step:

\[
\frac{1}{\Delta t} \begin{bmatrix} P^{i+1}_1 \\ P^{i+1}_2 \\ \vdots \\ P^{i+1}_{N-1} \\ P^{i+1}_N \end{bmatrix} = \begin{bmatrix} C^i_1 & D^i_1 & 0 & \cdots & 0 & 0 \\ B^i_2 & C^i_2 & D^i_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B^i_{N-1} & C^i_{N-1} & D^i_{N-1} \\ 0 & 0 & \cdots & 0 & B^i_N & C^i_N \end{bmatrix} \begin{bmatrix} P^i_1 \\ P^i_2 \\ \vdots \\ P^i_{N-1} \\ P^i_N \end{bmatrix},
\]

where everything on the right side is known. Therefore the solution can be obtained via a simple recursion of matrix multiplication.

3 Implementation

We use C++ and Python to implement the above model. Below is the pseudo code of model implementation.

**Algorithm 1:** callable/putable bond pricing method based on Hull-white model

**Input:** bond basic information, risk-free and risky yield curve, Hull-White parameters

**Output:** bond net value

1. Construct the time grid from last coupon payment date to maturity;
2. Obtain risk and risky continuous yield on each time grid point;
3. Calculate hazard rate \( \lambda(t) \) using formula (2.10);
4. Calibrate Hull-White parameter \( \vartheta(t) \) using formula (2.12);
5. Calculate bond price vector at option time \( T_m \) using formula (2.16);
6. Calculate option price vector at today \( T_0 \) using formula (2.17);
7. Get option price of interest rate \( r_0 \) using linear interpolation method;
8. Calculate bond straight price on today \( T_0 \) using formula (2.5);
9. Calculate bond net price:

\[
\text{net value} = \text{straight price} + \text{option price} - \text{accrued interest};
\]

4 Model’s Closed Form

4.1 Closed Form Derivation

First we calculate bond price, then the option price is derived.

4.1.1 Close Form of Bond Price

The equations (2.16) and (2.17) can be explicitly solved. Therefore, we can reduce computational complexity. The calculation is simple and listed below.
As we know, the bond with coupon payments can be seen as a sum of several zero-coupon bonds. Therefore, the bond price at option time $T_m$ can be written as follows:

$$P(T_m, r) = \sum_{j=m+1}^{n} CP(T_m, T_j, r) + FP(T_m, T_n, r), \quad (4.18)$$

where $T_j$ is the coupon payment time and $T_n$ is the maturity. $P(T_m, T_j, r)$ is the bond value at time $T_m$ which pays 1 dollar at maturity $T_j$ and satisfies the following differential equation:

$$\begin{cases}
\frac{\partial P}{\partial t} + (\varphi(t) - ar) \frac{\partial P}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial r^2} - (r + \lambda(t))P = 0, & T_m \leq t \leq T_j,

P(T_j, T_j, r) = 1.
\end{cases} \quad (4.19)$$

The above equation can be explicitly solved by assuming that:

$$P(T_m, T_j, r) = \exp(f_1(T_m) + f_2(T_m)r),$$

with boundary conditions: $f_1(T_j) = f_2(T_j) = 0$.

We can obtain the differentials of $P(T_m, T_j, r)$:

$$\frac{\partial P}{\partial t} = [f_1'(T_m) + f_2'(T_m)r]P, \quad \frac{\partial P}{\partial r} = f_2(T_m), \quad \frac{\partial^2 P}{\partial r^2} = f_2^2(T_m).$$

Substituting into equation (4.19), we have:

$$f_2'(T_m) - af_2(T_m) - 1 = 0, \quad f_1(T_j) = 0, \quad (4.20)$$

$$f_1'(T_m) + \varphi(T_m)f_2(T_m) - \lambda(T_m) + \frac{1}{2} \sigma^2 f_2^2(T_m) = 0, \quad f_2(T_j) = 0. \quad (4.21)$$

Thus

$$\begin{cases}
f_2(t) = \frac{\exp(-a(T_j - t)) - 1}{a}, \\
f_1(t) = \int_t^{T_j} [\varphi(s)f_2(s) - \lambda(s) + \frac{1}{2} \sigma^2 f_2^2(s)]ds.
\end{cases} \quad (4.22)$$

1. $\int_{T_m}^{T_j} \lambda(s)ds$ can be calculated by:

$$\int_{T_m}^{T_j} \lambda(s)ds = [Y^d(T_m, T_j) - Y^d(T_m, T_j)](T_j - t).$$

2. $\int_{T_m}^{T_j} \frac{1}{2} \sigma^2 f_2^2(s)ds$ can be calculated explicitly.

3. $\int_{T_m}^{T_j} \frac{1}{2} \varphi(s)f_2(s)ds$ can be calculated by numerical methods.
4.1.2 Close Form of Option Price

Next we calculate $O(t, r)$ based on the work of Hull and White (1990) and Jamshidian (1989). We present here the form of $P(T_m, r)$ firstly:

$$P(T_m, r) = \sum_{j=m+1}^{n} (C + \delta)P(T_m, T_j, r) + FP(T_m, T_n, r), \quad (4.23)$$

Since $f_2(t) \leq 0$, $P(T_m, r)$ is a decreasing function of $r$, an option on a portfolio of discount bonds is equivalent to a portfolio of options on the discount bonds with appropriate exercise prices. If there exists $r^*$ satisfying:

$$\sum_{m \leq j \leq n} (C + \delta)P(T_m, T_j, r^*) + FP(T_m, T_n, r^*) = K. \quad (4.24)$$

We can derive $r^*$ by numerical method.

$P(T_m, T_j, r)$ also decreases w.r.t. $t$. Therefore:

$$P(T_m, T_j, r) \geq P(T_m, T_j, r^*) \iff r \leq r^*.$$  

So, option payoff $O(T_m, r)$ is as follows:

$$O(T_m, r) = (P(T_m, r) - K)^+$$

$$= \sum_{m \leq j \leq n} (C + \delta)[P(T_m, T_j, r) - P(T_m, T_j, r^*)]^+ + F[P(T_m, T_n, r) - P(T_m, T_n, r^*)]^+$$

$$= \sum_{m \leq j \leq n} (C + \delta)O^j(T_m, r) + FO^n(T_m, r), \quad m \leq j \leq n$$

Let $O^j(t, r), m \leq j \leq n$ satisfy the following differential equation:

$$\begin{cases}
\frac{\partial O^j}{\partial t} + (\vartheta(t) - ar)\frac{\partial O^j}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 O^j}{\partial r^2} - (r + \lambda(t))O^j = 0, & t \in [0, T_m), \\
O^j(T_m, r) = [P(T_m, T_j, r) - P(T_m, T_j, r^*)]^+. &
\end{cases} \quad (4.25)$$

Then $O^j(t, r)$ can be viewed as an option on a zero coupon bond. The explicit form of $O^j(t, r)$ can be obtained by the distribution of $r(t)$. We denote $X_j = P(T_m, T_j, r^*), j = m, m+1, \cdots, n$. The form is showed in Hull and White(1990):

$$O^j(t, r) = P(t, T_j, r)N(b_i) - X_jP(t, T_m, r)N(b_j - \sigma_{pj}) \quad (4.26)$$

where:

$$\begin{cases}
b_j = \frac{1}{\sigma_{pj}} \log\left\{ \frac{P(t, T_j, r)}{P(t, T_m, r)X_j} \right\} + \frac{\sigma_{pj}}{2}, \\
\sigma_{pj} = [\exp(-aT_m) - \exp(-aT_j)]\sqrt{\frac{\sigma^2}{2\alpha^3}[\exp(2aT_m) - \exp(2at)]},
\end{cases} \quad (4.27)$$
and \( t \) is current time, \( T_j \) is the bond’s expire time, \( T_m \) is the option expire time, \( N(\cdot) \) is the cumulative normal distribution function.

According to work of Jamshidian (1989), the option on the coupon-bearing bond is therefore the sum of options on discount bonds with the exercise price of the \( j \)th option being \( O^j(t, r) \). Therefore we have:

\[
O(t, r) = \sum_{m \leq j \leq n} (C + \delta)O^j(t, r) + FO^n(t, r)
\]

### 4.2 Implementation Steps

The following paragraph presents the pricing steps of the embedded option in a callable bond at time \( t \). The interest rate at time \( t \) is \( r \). The features of the callable bond are described as in Dai and Xu (2014)’s work:

1. The face value is \( F \) and the maturity time is \( T_n \).
2. Strike on option date \( T_m \) is \( K \).
3. \( 0 < T_1 < T_2 < \cdots < T_m \) are the coupon dates. The coupon is denoted by \( C \) at every coupon date.
4. \( T_{m+1} < T_{m+2} < \cdots < T_n \) are the coupon dates. The coupon is denoted by \( C + \delta \) at every coupon date.

**step I**

The formula of the price of a zero coupon bond at \( t \) with final payment at \( T \) of \$1, i.e., \( P(t, T, r) \):

\[
P(t, T, r) = \exp(f_1(t) + f_2(t)r),
\]

where:

\[
\begin{align*}
    f_2(t) &= \exp(-a(T-t))^{-1}, \\
    f_1(t) &= \int_t^T [\vartheta(s)f_2(s) + \frac{1}{2}\sigma^2f_2^2(s) - \lambda(s)]ds,
\end{align*}
\]

1. \( \int_t^T \lambda(s)ds \) can be calculated by:

\[
\int_t^T \lambda(s)ds = \int_t^T \left[ Y^d(t, T) - Y^d(t, T)\right](T - t).
\]

2. \( \int_t^T [\frac{1}{2}\vartheta(s)f_2(s) + \frac{1}{2}\sigma^2f_2^2(s)]ds \) can be calculated by numerical methods.

**step II**
Solve \( r^* \) in the following equation by numerical method:

\[
\sum_{m \leq j \leq n} (C + \delta)P(T_m, T_j, r^*) + FP(T_m, T_n, r^*) = K.
\]

Remark: the left hand of the above equation is a decreasing function of \( r \).

We denote \( X_j = P(T_m, T_j, r^*), j = m, m+1, \ldots, n. \)

step III

According to work of [Jamshidian (1989)](http://example.com), the option on the coupon-bearing bond is the sum of options on discount bonds with the exercise price of the \( j \)th option being \( O_j(t, r) \).

\[
O(t, r) = \sum_{m \leq j \leq n} (C + \delta)O_j(t, r) + FO^n(t, r)
\]

- **For putable bond**
  
  The \( j \)th option’s formula is:

  \[
  O_j(t, r) = P(t, T_j, r)N(b_j) - X_jP(t, T_m, r)N(b_j - \sigma_{pj})
  \]

  where:

  \[
  \begin{aligned}
  b_j &= \frac{1}{\sigma_{pj}} \log \left\{ \frac{P(t, T_j, r)}{P(t, T_m, r)X_j} \right\} + \frac{\sigma_{pj}}{2}, \\
  \sigma_{pj} &= \left[ \exp(-aT_m) - \exp(-aT_j) \right] \sqrt{\frac{\sigma^2}{2a^3} \left[ \exp(2aT_m) - \exp(2at) \right]},
  \end{aligned}
  \]

  and \( N(\cdot) \) is the cumulative normal distribution function.

- **For callable bond**

  Similarly, the price of \( j \)th option \( O_j(t, r) \) at time \( t \) is:

  \[
  O_j(t, r) = P(t, T_j, r)N(b_j) - X_jP(t, T_m, r)N(b_j - \sigma_{pj}) - P(t, T_j, r) + X_jP(t, T_m, r)
  \]

  \[
  = X_jP(t, T_m, r)N(-b_j + \sigma_{pj}) - P(t, T_j, r)N(-b_j)
  \]

  where:

  \[
  \begin{aligned}
  b_j &= \frac{1}{\sigma_{pj}} \log \left\{ \frac{P(t, T_j, r)}{P(t, T_m, r)X_j} \right\} + \frac{\sigma_{pj}}{2}, \\
  \sigma_{pj} &= \left[ \exp(-aT_m) - \exp(-aT_j) \right] \sqrt{\frac{\sigma^2}{2a^3} \left[ \exp(2aT_m) - \exp(2at) \right]},
  \end{aligned}
  \]

  and \( N(\cdot) \) is the cumulative normal distribution function.
References


