Perpetual Bonds Valuation: Documentation

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Abstract
In this documentation, more details will be given for 273 perpetual bonds. Variables used in our valuation model and how those variables were obtained from raw data will be scrutinized, to ensure data accuracy and to anticipate future correction or validation of our database. More rigorous discussions and evaluations on current methodologies used will be presented.

Keywords: Perpetual Bonds, Step-up Rate, Coupon Adjustment, Redemption, Simulation, Default Risk
1. **Our Milestones**

Here are our milestones:

1. In mid-August, SL, ST, and TJ discussed about implementation of simulation method initiated by ST.
2. In end-August to beg-September, SL implemented the simulation for perpetual bonds using MATLAB and Python.
3. In beg-Sept, TJ uploaded relevant variables to database, by reading from clauses in Wind.
4. In beg-Sept to mid-Sept, TJ implemented the PDE method for perpetual step-up bonds using Python.
5. In beg-Sept to mid-Sept, SL and TJ jointly improved the Python code to be integrated to database (i.e., automatic daily input from database and upload to database).
6. In end-Sept, discussion about current simulation method is done by RF, SL, ST, TJ.
7. In end-Sept to beg-Oct, SL wrote documentation, including proposed new methods for valuation.
8. In mid-Oct and end-Oct, ST and TJ gave feedbacks about this documentation.

2. **Introduction**

In our database `rmi_bond_basic`, 1182 bonds are given binary value 'is_option', 'is_floating', and 'is_perpetual', corresponding to whether the bond has options, whether the bond has floating coupon part, and whether the bond has no upper bound for maturity. The classification could be summarized by:

<table>
<thead>
<tr>
<th>is_option</th>
<th>is_floating</th>
<th>is_perpetual</th>
<th>Category of Bonds</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>Perpetual Floating</td>
<td>261</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>Perpetual Step-up</td>
<td>12</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>Putable/Callable</td>
<td>651</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>Floating Rate</td>
<td>258</td>
</tr>
</tbody>
</table>

Furthermore, perpetual floating rate bonds could be classified by its floating benchmark: SHIBOR or government bond yield. There are 2 bonds with
SHIBOR as floating benchmark, and there are 259 bonds with government bond yield as floating benchmark.

In this documentation, we focus on pricing perpetual step-up bonds and perpetual floating rate bonds. For all bonds in these categories, the bond could be bought back by the issuer on predetermined date with redemption price equal to face value. The coupon is paid annually at the interest accrual date of each year.

3. Necessary and Sufficient Information, Data Validation

Given unique Wind ID of a bond, these eleven variables are static (time-independent):

1. Chinabond spot yield curve code (classified by type and rating), like '1101', '1431', defined by Chinabond.
2. Face value, usually 100.
3. Issuance date
4. Initial benchmark rate, which is the floating part of first decided coupon rate. Note that this variable is not relevant and not available for perpetual step-up.
5. Coupon rate at issuance, which is the decided coupon rate at issuance.
6. Step-up rate, in basis points, like 300 bps, 400 bps, 600 bps.
7. Stability of step-up rate, which is of binary value.
8. First coupon adjustment, integer (in years) counting from issuance.
9. Frequency of coupon adjustment, integer (in years).
10. First redemption date, integer (in years) counting from issuance.
11. Redemption frequency, integer (in years).

* more on 6 & 7 will be explained in Section 4.

In addition to 11 static variables, some dynamic (time-dependent) variables:

1. Transaction date,
2. n-year spot yields at transaction date (n may take several discrete values, say 0.25, 0.5, 1, 2, 3, 5, 10), which correspond to Chinabond yield curve code, provided by Chinabond,
3. n-year spot yields at transaction date (\( n \) may take several discrete values, say 0.25, 0.5, 1, 2, 3, 5, 10), which correspond to government/SHIBOR benchmark yield curve, provided by Chinabond, are needed to do valuation of a bond on transaction date in our current program, with our current model.

Here are short descriptions on how the variables are obtained from raw data:

1. **Chinabond spot yield curve code**: Use Excel function (with Wind add-ins) `yccode` given wind ID to get 4-digit Wind curve code. However, change the last digit '2' to '1' (since we use spot yield rather than yield to maturity). These data are then uploaded to column `yc_id` datasets `rmi_bond_basic` and `rmi_bond_info` by Computer Support Team.

2. **Face value, Issuance date, Initial Benchmark Rate, Coupon rate at Issuance, First Redemption Date, \( n \)-year spot yields at transaction date which correspond to government/SHIBOR (choose one) and Chinabond yield curve code**: Computer Support Team extracted these variables (with other variables) from Wind by Python code referring to Python code generator, then upload these directly to the database. These variables correspond to:
   - 'face_value' in `rmi_bond_info` and `rmi_bond_listview`,
   - 'carry_date' in `rmi_bond_info` and `rmi_bond_listview`,
   - 'base_rate_init' in `rmi_bond_floating`,
   - 'coupon_rate' in `rmi_bond_info` and `rmi_bond_listview`,
   - 'redemption_date' in `rmi_bond_option` and `rmi_bond_listview`,
   - 'val_x' and 'val_y' in `rmi_yc_spot`, respectively.

3. **Step-up rate, stability of step-up rate, first coupon adjustment, frequency of coupon adjustment, redemption frequency**: Currently, human intelligence is needed to obtain these numbers from reading the clauses in Wind Data Explorer (Bonds \( \rightarrow \) Basic Info \( \rightarrow \) Floating Rate Debt Elements \( \rightarrow \) Benchmark Determining Method & Bonds \( \rightarrow \) Basic Info \( \rightarrow \) Debt with Embedded Option Elements \( \rightarrow \)
Figure 1: Callable option times

Special Provisions). These variables are then put in Excel and uploaded to columns 'step_up', 'cpn_stable', 'first_cpn_adj_time', 'cpn_adj_freq', 'redemption_freq', respectively in rmi_bond_perpetual.

We should take note the importance of making sure that all variables are accurate and consistent. Therefore, regular validation is needed for 11 static variables by Modelling Team. To do the validation, one of the method is: Get 11 corresponding variables for 273 bonds from raw data (either using Excel function, using Python code to extract data from Wind, or reading clauses), then compare with those in the database.

4. Product Descriptions

Given a perpetual step-up or perpetual floating rate bond. In this section, we try to give description of the bond. Recall that in Section 3, these are some variables of a bond: First coupon adjustment, Frequency of coupon adjustment, First redemption date, Redemption frequency. For simplicity, we denote it in this section as $FA$, $QA$, $FR$, and $QA$, respectively. Note that all are integers. Also, define initial spread to be Coupon rate at issuance minus Initial benchmark rate.

Basic Information
The bond is issued at Issuance date with certain Face value. At transaction date, the bond term structure is assumed to follow Spot yields correspond to Chinabond yield curve code (classified by type and rating).

Callable Options
The bond could be bought back by the issuer at the end of year $FR$, $FR+QR$, $FR+2QR$, $FR+3QR$, ···. Since it coincides with coupon payment time, it is assumed that coupon is paid just before the option is exercised (in other words, if option is exercised at that time, the issuer should pay redemption price plus coupon at that time). Refer to Figure 1 for illustration.

Coupon Payments
The coupons are paid annually at the end of years. Let $CR$ be the annual
coupon rate before the adjustment (which is just equal to **Coupon rate at issuance**), and let \( CR_k \) be the annual coupon rate decided (adjusted) at time \( FA + kQA \) for \( k \geq 0 \). Here are the summary of coupon payment:

<table>
<thead>
<tr>
<th>Coupon adjustment time</th>
<th>Coupon payment interval</th>
<th>Coupon rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( (0, FA] )</td>
<td>( CR )</td>
</tr>
<tr>
<td>( FA )</td>
<td>( (FA, FA + QA] )</td>
<td>( CR_0 )</td>
</tr>
<tr>
<td>( FA + QA )</td>
<td>( (FA + QA, FA + 2QA] )</td>
<td>( CR_1 )</td>
</tr>
<tr>
<td>( FA + 2QA )</td>
<td>( (FA + 2QA, FA + 3QA] )</td>
<td>( CR_2 )</td>
</tr>
</tbody>
</table>

We will go on with more details on \( CR_k \) for three types of perpetual bonds: perpetual step-up, perpetual floating SHIBOR, and perpetual floating government yield. Denote the **Step-up rate** by \( s \).

### 4.1. Perpetual Step-Up

Since initial benchmark rate is not relevant, the initial spread is just equal to coupon rate at issuance and is equal to \( CR \). Actually the stability of step-up rate is always False (unstable step-up rate) in our current bond list, but for the sake of completeness, here are the descriptions:

- If the step-up is unstable, \( CR_k \) is equal to \( CR + (k + 1)s \).
- If the step-up is stable and \( s \neq 0.06 \), \( CR_k \) is equal to \( CR + s \).
- If the step-up is stable and \( s = 0.06 \), \( CR_0 = C + 0.03 \) and \( CR_k \) is equal to \( CR + 0.06 \) for \( k > 0 \).

### 4.2. Perpetual Floating SHIBOR

There are two perpetual floating SHIBOR: ’101555025.IB’ and ’101682002.IB’. Actually these bonds have stable step-up with \( s = 0.06 \). But for the sake of completeness, we give general descriptions:

- If the step-up is unstable, \( CR_k \) is equal to initial spread plus \( (k + 1)s \) plus 750-days average of \( QA \)-year SHIBOR rate for 750 days preceeding coupon adjustment time (year \( FA + kQA \)).

- If the step-up is stable and \( s \neq 0.06 \), \( CR_k \) is equal to initial spread plus \( s \) plus 750-days average of \( QA \)-year SHIBOR rate for 750 days preceeding coupon adjustment time (year \( FA + kQA \)).
If the step-up is stable and \( s = 0.06 \), \( CR_0 \) is equal to initial spread plus 0.03 plus 750-days average of QA-year SHIBOR rate for 750 days preceding coupon adjustment time (year \( FA \)), and for \( k > 0 \), \( CR_k \) is equal to initial spread plus 0.06 plus 750-days average of QA-year SHIBOR rate for 750 days preceding coupon adjustment time (year \( FA + kQA \)).

4.3. Perpetual Floating Government Yield

Exceptions are made for '101555033.IB' and '101564063.IB'. Here are the general descriptions:

- For perpetual floating government yield with unstable step-up, \( CR_k \) is equal to initial spread plus \((k + 1)s\) plus QA-year government yield at 5 trading days before the coupon adjustment time (year \( FA + kQA \)).

- For perpetual floating government yield with stable step-up and \( s \neq 0.06 \), \( CR_k \) is equal to initial spread plus \( s \) plus QA-year government yield at 5 trading days before the coupon adjustment time (year \( FA + kQA \)).

- For perpetual floating government yield with stable step-up and \( s = 0.06 \), \( CR_0 \) is equal to initial spread plus 0.03 plus QA-year government yield at 5 trading days before the coupon adjustment time (year \( FA \)), and for \( k > 0 \), \( CR_k \) is equal to initial spread plus 0.06 plus QA-year government yield at 5 trading days before the coupon adjustment time (year \( FA + kQA \)).

- For '101555033.IB', \( CR_0 \) is equal to initial spread plus 0.03 plus 750-days average of QA-year government yield for 750 trading days preceding the coupon adjustment time (year \( FA \)), and for \( k > 0 \), \( CR_k \) is equal to initial spread plus 0.06 plus 750-days average of QA-year government yield for 750 trading days preceding the coupon adjustment time (year \( FA + kQA \)).

- For '101564063.IB', \( CR_k \) is equal to initial spread plus 0.03 plus average of 5-year government yield and 7-year government yield at 5 trading days before coupon adjustment time (year \( FA + kQA \)). Note that for this bond, \( QA = 6 \), and average is taken because 6-year government yield is not published.
5. Current Methodologies

Our current code could do valuation for 12 perpetual step-up bonds and 257 non-exceptional perpetual floating government yield bonds. We use PDE method to do valuation for 12 perpetual step-up bonds, and we use simulation method to do valuation for 257 non-exceptional perpetual floating government yield bonds. Actually, we could use simulation method to do valuation on 12 perpetual step-up bonds, just to compare with PDE method. We will discuss each methodologies in subsections. We first take note of the following Lemma:

**Lemma:** Suppose that for $s_1 \leq t \leq s_2$,

$$V(r_t, t) = \mathbb{E}_Q \left[ \exp \left( - \int_{t}^{s_2} (r_u + d(u))du \right) h(r_{s_2}) \bigg| F_t \right]$$

where $Q$ is risk-neutral measure, $r_t$ is the only random variables and follows Hull-White model ($dr_t = (\vartheta(t) - ar_t)dt + \sigma dW_t$), $d$ is deterministic time-dependent, and $h(r)$ is given continuous function. Then

$$\frac{\partial V}{\partial t} + (\vartheta(t) - ar) \frac{\partial V}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial r^2} - (r + d(t))V = 0$$

for $s_1 \leq t \leq s_2$ with boundary conditions $V(r_{s_2}, T) = h(r_{s_2})$.

5.1. PDE

Here are the steps:

1. We assume the bond’s maturity to be $v$-th option time (it is bought back by the issuer not later than $v$-th option time). We choose $v$ to be 3, 4, or 5. The convention for time is: 0 denotes current time and T denotes maturity, in years. We also introduce the concept of left-time $t-$, for which the coupon and option (if exists) at time $t$ is paid after $t-$ and before $t$. If there is coupon at $t$, denote it by $C_t$, and if there is option at time $t$, denote it by $K_t$ (they are deterministic).

2. Using defaultable yield curve (i.e., smooth-spline interpolated yield curve corresponding to Chinabond yield curve code classified by type and rating) and default-free yield curve (i.e. smooth-spline interpolated yield curve from Chinabond’s ’1231’ yield data), we could get estimate of deterministic time-dependent $\lambda$ (hazard rate) and $\vartheta$ (mean-reversion level rate) as in Xu Jing’s work. Assume $a = 12.1019$ and $\sigma = 0.0328$. 


as Hull-White parameters. We assume that short rate $r$ of default-free follows Hull-White model.

3. Denoting the adapted process $r$ by $r_t$ at time $t \geq 0$, we will show that the price of the bond at time $0 \leq t \leq T$ depends only on $t$ and $r_t$ at $t$ (it’s independent of previous path of $r$ before $t$!), by backward induction on time from $T$ to 0. By convention of time, $r_{t-} = r_t$. To use induction, we take not of the following facts:

- Price of bond at time $T$, $P(r_T, T)$, is zero.

- If there is option with strike $K_t$ or coupon $C_t$ at time $t$ (recall the convention of time), assuming price of bond at $t$, $P(r_t, t)$ depends only on $r_t$, then the price of the bond at time $t-$, is equal to

$$
\begin{cases}
P(r_t, t-) = P(r_t, t) + C_t & \text{if there is coupon and no option at } t \\
P(r_t, t-) = \min(P(r_t, t), K_t) & \text{if there is option and no coupon at } t \\
P(r_t, t-) = \min(P(r_t, t), K_t) + C_t & \text{if there is coupon and option at } t
\end{cases}
$$

which indeed depends only on $r_t$.

- If $t_1 \leq t_2$ be such that there is no option/coupon on interval $[t_1, t_2-]$, assuming price of the bond at $t_2-$ say $P(r_{t_2}, t_2-)$ depends only on $r_{t_2}$, then for $t_1 \leq t \leq t_2-$ price of the bond at time $t$ is equal to

$$
\mathbb{E}^Q \left[ e^{\int_{t_1}^{t_2} (r_s + \lambda(s)) ds} P(r_{t_2}, t_2-) \bigg| F_t \right],
$$

meaning that price at $t$ indeed depends only on $r_t$.

The first fact serves as initial step and the second and third step serve as inductive steps.

4. Using Lemma, we could summarize the information in (3) by:

$$
\begin{cases}
P(r_T, T) = 0 \\
P(r_t, t-) = P(r_t, t) + C_t & \text{if there is coupon and no option at } t \\
P(r_t, t-) = \min(P(r_t, t), K_t) & \text{if there is option and no coupon at } t \\
P(r_t, t-) = \min(P(r_t, t), K_t) + C_t & \text{if there is coupon and option at } t \\
\frac{\partial P}{\partial t} + (\vartheta(t) - ar) \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} - (r + \lambda(t)) P = 0 & \text{for } t_1 \leq t \leq t_2 \\
\end{cases}
$$

\[
\text{with } t_1 \leq t_2 \text{ be such that there is no option/coupon on interval } [t_1, t_2-]
\]
5. Therefore, given current short rate $r_0$, we could obtain current price $P(r_0, 0)$ by solving the PDE. Currently, we use finite difference method to solve the PDE, and use linear interpolation at $r_0$, given $\{P(w, 0) | w \in W\}$, where $W$ is set of $r_0$'s such that $P(w, 0)$ is numerically computed using finite difference method, to obtain price $P(r_0, 0)$. Refer to Figure 2 for illustration.

5.2. Simulation

For simplicity, we assume that at coupon adjustment time, yield on that day is used instead of 5 trading days before to decide future floating coupon rate.

1. We assume the bond’s maturity to be $v$-th option time (it is bought back by the issuer not later than $v$-th option time). We choose $v$ to be 3, 4, or 5. The convention for time is: 0 denotes current time and $T$ denotes maturity, in years. Construct time grid $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ such that $\{t_i | 0 \leq i \leq n\}$ contains coupon payment time, coupon adjustment time, and option time. If there is option at
Figure 3: Simulation of short rate paths using Hull-White model

$t_j$, denote it by $K_j$. Note that the coupon has fixed (predetermined at time 0) part and floating part (determined later).

2. Using defaultable yield curve (i.e., interpolated yield curve corresponding to bond’s Wind curve code classified by type and rating) and default-free yield curve (i.e. interpolated yield curve from Wind’s 1231 yield data), we could get estimate of deterministic time-dependent $\lambda$ (default rate) and $\vartheta$ (long-run short rate). Assume $a = 12.1019$ and $\sigma = 0.0328$ as Hull-White parameters. We assume that short rate $r$ of default-free follows Hull-White model.

3. The floating coupon part looks like this: Coupon at time $t_j$ is determined at from $QA$-year government yield at time $t_i$, for some $i < j$. Note that since we take government yield as risk-free yield, $QA$-year government yield is just equal to yield of expected risk-free zero-coupon bond price at time $t_i$ maturing at $T$, say $P_{hw}(r_{t_i}, t_i, t_i + QA)$ computed by Hull-White formula.

4. Generate a path of short rates using Hull-White model $r_0, r_1, r_2, \ldots, r_n$ corresponding to time $t_0, t_1, t_2, \ldots, t_n$. Note that $r_0$ is current short
rate, and the steps are

\[ r_{i+1} - r_i = (\vartheta(t_i) - ar_i)(t_{i+1} - t_i) + \sigma \sqrt{t_{i+1} - t_i} Z \]

where \( Z \) is standard normal random variable generated. Refer to Figure 3 for 1000 generated paths of short rate \( r_i \).

5. Note that the path generated in (3), floating coupon part \( C_j \) is realized at \( t_i \) for some \( i < j \) using simulated short rate \( r_i \) and Hull-White formula. Therefore, we could add fixed coupon and floating coupon parts after simulating the short-rate path. Denote the total coupon payment at time \( t_i \) to be \( C_i \), if exists.

6. We calculate current price \( P(0) \) corresponding to a short-rate path by discounting. Let \( P(t_i) \) be price at \( t_i \). Here are the steps:

- \( P(t_n) = C_n + K_n \)
- Given \( P(t_{i+1}) \), then \( P(t_i) \) is equal to

\[
\begin{cases} 
  C_i + \min \left( P(t_{i+1})D(t_i, t_{i+1}), K_i \right) & \text{if there is coupon and option at } t_i \\
  C_i + P(t_{i+1})D(t_i, t_{i+1}) & \text{if there is coupon but no option at } t_i \\
  \min \left( P(t_{i+1})D(t_i, t_{i+1}), K_i \right) & \text{if there is option but no coupon at } t_i 
\end{cases}
\]

where \( D(t_i, t_{i+1}) = 1/exp((\lambda(t_i) + r_i)(t_{i+1} - t_i)) \) is the discounting factor including the hazard rate.

Note that the discounting factor involves \( \lambda \) because the probability of default between \( t_i \) and \( t_{i+1} \) is approximately \( exp(\lambda(t_i)(t_{i+1} - t_i)) \).

7. Repeating steps 4 to 6 many times, say 1000 times, we obtain 1000 value of \( P(0) \)'s. Taking average, we obtain the expected current price of the bond in our simulation. Taking sd of \( P(0) \)'s, we obtain expected sd of the bond price \( P_0 \), assuming only uncertainty of short rate and not default event. Subtract with accrued interest to obtain current clean price.

5.3. Discussions

We assume maturity at third, fourth, or fifth option, by looking at lifetime of previously called bond, mostly called on first option. Although the maturity assumption fits with this observations, theoretical justification is needed, i.e., if we denote \( P_v \) to be theoretical price, assuming maturity to be
v-th option, will the sequence \( P_1 > P_2 > P_3 > \cdots \), how big is \( v \) such that \( P_v \) is close enough to true perpetual bond model price \( P = \lim_{v \to \infty} P_v \)?

In simulation, further justification is needed on the convergence of price for simulation of \( N \) short-rate paths, i.e., how big \( N \) should be given certain error of price we tolerate? Note that for grid length of 0.01 years, we generate \([100 \times \text{maturity (in years)}]\) standard random normal for each path, which is very large.

The use of Kalman Filter to adjust model price with transaction price is currently postponed, caused by lack of theoretical justifications.

6. Current Code Methodology

The structure of our current code is like this:

1. for set of perpetual floating government bond yield, get static variables correspond to each bond code:
   for set of transaction date, get dynamic variables correspond to each transaction date:
       run the simulation to obtain model price
2. for set of perpetual step-up bond, get static variables correspond to each bond code:
   for set of transaction date, get dynamic variables correspond to each transaction date:
       run the PDE to obtain model price

Current results are stored in `rmi_bond_model` from 2013.

7. Possible Direction of Our Project

It is possible to implement alternative methods proposed and compare the results (see Appendix), which will improve the computational accuracy, yet will not affect valuations too much because the option part is small compared to straight price. What should be noticed is that our current model has some serious flaws. Firstly, the estimation of credit risk (default risk) is very raw: We use sectionalized yield curve based on Wind classification, calculate the spread with risk-free yield curve, to obtain the default intensity rate. In the future, we could try to implement structural model, taking into account
the financial information about the company and its sector to model the default risk. Moreover, we could find risks for specific product by learning the motivation to issue perpetual, callable, putable bonds.

Besides this, we could also consider the tax effect. Since government yield is not taxable and most of other bonds are taxable, there will be tax spread.

8. Bibliography


9. Appendix: Proposed New Methodologies for Perpetual Floating Chinabond

9.1. Multi-PDE

For simplicity, we assume that at coupon adjustment time, yield on that day is used instead of 5 trading days before to decide future floating coupon rate. Let 0 be current time, $T$ be maturity, assumed to be $v$-th option, in years. We also introduce the concept of left-time $t-$, for which the coupon and option (if exists) at time $t$ is paid after $t-$ and before $t$.

Let $0 < t_1 < t_2 < \cdots < t_n = T$ be coupon payment time. Since option time and coupon adjustment time always coincide with coupon payment time, if there is option at $t_i$, denote the strike price by $K_i$, and let coupon adjustment time be $0 < t_{a_1} < t_{a_2} < \cdots < t_{a_m} < T$, where $0 < a_1 < a_2 < \cdots < a_m < n$ are integers. Note that the coupon rate adjusted at $t_{a_j}$ is effective for coupon at $t_i$ for $a_j < i \leq a_{j+1}$ (assume $a_0 = 0$ and $a_{m+1} = n$).

Let $r_t$ be adapted short-rate process following Hull-White model with $a = 12.1019$ and $\sigma = 0.0328$. For time $0 \leq t \leq T$, let $g(t) = t_{a_i}$ and $h(t) = t_{a_{i+1}}$ for some $i$ such that $t_{a_i} < g(t) \leq t_{a_{i+1}}$.

We will show by induction that price of bond at time $t$ depends only on $r_t$ and $r_{g(t)}$: This means that price of bond at time $t$ only depends on short rate at $t$ and short rate at previous coupon adjustment time (or current time if there is not any). These facts are used in backward induction on time:

- Price of bond at time $T$, $P(r_T, T, r_{t_{am}})$, is zero.
Figure 4: Illustration for hypothetical bond with maturity 21 years and coupon adjustment every 3 years. In this case $a_i = 3i$, $g(4) = 3$, $g(11) = 9$. The PDE is "back-forth" between two bold lines.
inductive steps. The first fact serves as initial step and the second and third step serve as
P\texttt{r}\texttt{ent current clean price. Refer to Figure 4 for illustration.}
P\texttt{terpolation to obtain }P\texttt{then continue to move backward to obtain }P\texttt{until we finally obtain }
P\texttt{tion, equations (2), (3), (4) interchangebly for PDE and jump conditions}
to find
The PDE is quite complicated, but here are the rough guides: We first aim
\left\{
\begin{array}{ll}
P(r_t, t_i^-, r_{g(t_i)}) = P(r_t, t_i, r_{g(t_i)}) + C_i & \text{if there is no option on } t_i \\
P(r_t, t_i^-, r_{g(t_i)}) = \min(P(r_t, t_i, r_{g(t_i)}), K_i) + C_i & \text{if there is option on } t_i
\end{array}
\right.
which indeed only depend on }r_t\text{ and } r_{g(t_i)}\text{ since } C_i \text{ depends only on } \r_{g(t_i)}.

For } g(t_i) < t_i \leq h(t_i)\text{, and } t_{i-1} \leq t \leq t_i\text{ (no option/coupon in this interval), assuming that the price of bond at } t_i^-\text{ depends only on } r_t\text{ and } r_{g(t_i)}, \text{ then the price of bond at time } t \text{ is }
\begin{equation}
\mathbb{E}_{\mathcal{Q}}\left[ \exp\left( - \int_{t}^{t_i} (r_s + \lambda(s))ds \right) P(r_t, t_i^-, r_{g(t_i)}) \mid F_t \right],
\end{equation}
depending only on }r_t\text{ and } r_{g(t_i)} = r_{g(t)}.
The first fact serves as initial step and the second and third step serve as inductive steps.

Moreover, by Lemma, we obtain the following PDE and jump conditions:
\begin{equation}
\begin{cases}
P(r_T, T, r_{t_{am}}) = 0 \\
P(r_t, t^-, r_{g(t)}) = P(r_t, t, r_{g(t)}) + C_t & \text{if there is coupon and no option at } t \\
& \text{and } t_{a_j} < t \leq t_{a_{j+1}}, 0 \leq j \leq m \\
P(r_t, t^-, r_{g(t)}) = \min(P(r_t, t, r_{g(t)}), K_t) + C_t & \text{if there is coupon and option at } t \\
& \text{and } t_{a_j} < t \leq t_{a_{j+1}}, 0 \leq j \leq m \\
\frac{\partial P}{\partial t} + (\vartheta(t) - ar)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} - (r + \lambda(t))P = 0 & \text{for } t_{i-1} \leq t \leq t_i^- \\
& \text{where } r_{g(t)} \text{ is fixed and the partial differentiations are with respect to } r_t \text{ and } t.
\end{cases}
\end{equation}
The PDE is quite complicated, but here are the rough guides: We first aim to find } P(r_{t_{am}}, t_{am}, r_{t_{am}}) \text{ for a given } r_{t_{am}}. \text{ Use equation (1) for end condition, equations (2), (3), (4) interchangebly for PDE and jump conditions until we finally obtain } P(r_{t_{am}}, t_{am}, r_{t_{am}}). \text{ Repeat this for different } r_{t_{am}}'s, \text{ then continue to move backward to obtain } P(r_{t_{am-1}}, t_{am-1}, r_{t_{am-1}}) \text{ for different } r_{t_{am-1}}'s, P(r_{t_{am-2}}, t_{am-2}, r_{t_{am-2}}) \text{ for different } r_{t_{am-2}}'s, \cdots \text{ until we obtain } P(r_0, 0, r_0) \text{ for different } r_0's. \text{ Given current short rate } r_0, \text{ we use linear interpolation to obtain } P(r_0, t, r_0). \text{ Subtract with accrued interest to obtain current clean price. Refer to Figure 4 for illustration.}
The advantage of this method is that it doesn’t involve sampling error (such as in simulation). The error only possibly come from computation of PDE by finite difference method. However, the computation may take longer time because the PDE is 3-dimensional.

9.2. Semi Simulation

One of the problem for simulation is the convergence. For each path, if we use grid length of 0.01 years, then we generate \([100 \times \text{maturity (in years)\]}\] standard random numbers to simulate a short rate path. Therefore, the estimated price for each path depend on the randomness of these set of random numbers. In semi-simulation method, we try to reduce the number of random numbers generated for each path.

Firstly, we recall this fact: Suppose that \(r\) follows Hull-White. Given \(r_t\) and \(T > t\), random variables \(r_T\) and \(\int_t^T r_s ds\) are bivariate normal distribution. Let the sequence \((r_{t_0}, r_{t_1}, r_{t_2}, \ldots, r_{t_n})\) be sequence of short rate at time \(0 = t_0 < t_1 < \cdots < t_n\) where \(t_0\) denotes the current time and \(t_i\) denotes the time where there is option/coupon/coupon adjustment. We generate random numbers \((\int_{t_0}^{t_1} r_s ds, r_{t_1}, \int_{t_1}^{t_2} r_s ds, r_{t_2}, \ldots, r_{t_{n-1}}, \int_{t_{n-1}}^{t_n} r_s ds, )\), which follows the steps as:

- Given current short rate \(r_0\), then \((\int_{t_0}^{t_1} r_s ds, r_{t_1})\) are bivariate normal.
- Given short rate \(r_{t_1}\), then \((\int_{t_1}^{t_2} r_s ds, r_{t_2})\) are bivariate normal.
- etc ...

Therefore we involve only \(2n - 1\) random numbers, to obtain one simulated price, which is some function of these random numbers. Taking average of price realized from those random variables, we obtain expected price.

The advantage of this method is, theoretically, it converges faster than previous simulation method with respect to number of simulated paths, since it involves less random variables.

Remark 1: An alternative methods to improve the speed of multi-PDE, using Gaussian process when taking random sample of short-rate and short-rate at previous coupon adjustment time, is being discussed with Ruofei. This serves as a balance between simulation (faster time, but less justification of convergence) and multi-PDE (longer time, exact justification of price, but with error in numerical computation).
Remark 2: For all of these methods, the convergence with respect to assumed maturity is still not justifiable, although empirically, most of the bonds are called near the first option time.